evaluate the set of sufficient conditions (11) for uncoupled estimation:

$$P_{xx} = \sum_{i=1}^{M} R^2 \cos^2\left[\frac{2\pi(i-1)}{M}\right] + \sum_{i=M+1}^{2M} R^2 \cos^2\left[\frac{2\pi(i-M-1)}{M}\right] = R^2 M$$

$$P_{yy} = \sum_{i=1}^{M} R^2 \sin^2\left[\frac{2\pi(i-1)}{M}\right] + \sum_{i=M+1}^{2M} R^2 \sin^2\left[\frac{2\pi(i-M-1)}{M}\right] = R^2 M$$

$$P_{xy} = \sum_{i=1}^{M} R^2 \cos\left[\frac{2\pi(i-1)}{M}\right] \sin\left[\frac{2\pi(i-1)}{M}\right] + \sum_{i=M+1}^{2M} R^2 \cos\left[\frac{2\pi(i-M-1)}{M}\right] \times \sin\left[\frac{2\pi(i-M-1)}{M}\right] = 0$$

$$P_{xz} = \sum_{i=1}^{M} RZ \cos\left[\frac{2\pi(i-1)}{M}\right] - \sum_{i=M+1}^{2M} RZ \cos\left[\frac{2\pi(i-M-1)}{M}\right] = 0$$

$$P_{yz} = \sum_{i=1}^{M} RZ \sin\left[\frac{2\pi(i-1)}{M}\right] - \sum_{i=M+1}^{2M} RZ \sin\left[\frac{2\pi(i-M-1)}{M}\right] = 0.$$

The conditions given in (11) are therefore satisfied, and we conclude that the estimates of azimuth and elevation angles made with this two-ring array are uncoupled. It follows that an array consisting of several ring pairs, where each pair in (12) can have distinct values of Z and R, also has the property that the angle estimates are uncoupled. Cylindrical arrays and spherical arrays are examples of such arrays.

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A Simple Algorithm for Generating Discrete Prolate Spheroidal Sequences

Don M. Gruenbacher and Donald R. Hummels

***Abstract*— The discrete prolate spheroidal sequences are optimum waveforms in many communication and signal processing applications because they comprise the most spectral efficient set of orthogonal sequences possible. Generation of the sequences has proven to be difficult in the past due to the absence of a closed form solution. A new method of easily generating any single discrete prolate spheroidal sequence, including sequences of very long length, is presented. Also shown are some example sequences generated using the algorithm presented.**

I. Introduction

The set of discrete prolate spheroidal sequences (DPSS's) has long been known to have characteristics which would make them ideal candidates for use in communication and signal processing applications. Possible applications of DPSS's include pulse shaping, secure communications [1], PAM [2], and signal extrapolation [3]. Initially, DPSS generation was difficult due to the lack of a closed form solution. In [4] a method was given for generating the single most frequency-compact sequence out of a set of DPSS's, however, the method did not allow for the generation of either DPSS's beside the most compact sequence or for DPSS's of extremely long length. The recursive procedure in [4] does not work well for certain DPSS's with a length beyond approximately 32 samples. Applications requiring sequences with long lengths would not be able to use DPSS pulses in these cases. This correspondence will give a fairly straightforward, simple, and reliable method for the generation of any DPSS of arbitrary length. DPSS's with length in excess of 10 000 samples have been generated with this method. First, a description of the DPSS's and some of their properties will be given.

DPSS's are the discrete-time relatives of the prolate spheroidal wave functions [5] and discrete prolate spheroidal wave functions. Although DPSS's have been studied by various authors, the most extensive work has been done by Slepian [6]. The notation used by Slepian in denoting a single DPSS is $v^{(k)}(N, W)$, where N indicates both the number of DPSS's in the set and the number of samples in each DPSS, and W is a bandwidth and shaping factor under the constraint $0 < W < .5$. The value of k indicates the particular DPSS out of the set $k = 0, 1, \ldots, N-1$.

The most well known property of the orthogonal DPSS's is that a set of DPSS's comprise the index limited sequences with greatest amount of energy contained within a frequency band. It is also known that DPSS $v^{(0)}(N, W)$ has the most compact spectrum out of the set, while each succeeding DPSS, $k = 1, \ldots, N-1$, has a larger bandwidth than the DPSS preceding it [6].

II. DPSS Generation

In [6] Slepian presented properties of DPSS's which can be used to develop different techniques for generating any particular set of DPSS's. The two primary techniques find the DPSS's as eigenvectors of one or the other of two matrices. Matrix $H(N, W)$ is Hermitian,

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and its elements are defined by

$$H(N,W)_{nm} = \frac{\sin 2\pi W(m-n)}{\pi(m-n)}, \quad m,n = 0,1,\cdots,N-1. \tag{1}$$

The symmetric tridiagonal matrix $T(N,W)$, is defined by

$$T(N,W)_{ij} = \begin{cases} \frac{1}{2}i(N-i), & j = i-1 \\ \left(\frac{N-1}{2}-i\right)^2 \cos 2\pi W, & j = i \\ \frac{1}{2}(i+1)(N-1-i), & j = i+1 \\ 0, & |j-i| > 1 \end{cases} \tag{2}$$

where

$$i,j = 0,1,\ldots,N-1.$$

Matrix $T(N,W)$ is a normal symmetric Jacobian matrix. It can be shown that $T(N,W)$ commutes with $H(N,W)$, i.e.

$$H(N,W)T(N,W) = T(N,W)H(N,W) \tag{3}$$

so all the eigenvectors of $T(N,W)$ are also the eigenvectors of $H(N,W)$ [7].

In previous work [8] the QR algorithm was used to find the eigenvectors (DPSS's) of both $T(N,W)$ and $H(N,W)$. As expected, the same results were obtained from each matrix for small N. However, for $N > 32$, $H(N,W)$ becomes ill-conditioned and its computed eigenvectors are no longer the true DPSS's. Thus, only matrix $T(N,W)$ will provide reliable DPSS generation for large N. The only disadvantage of using $T(N,W)$ is that its eigenvalues do not indicate true bandwidth concentrations of the DPSS's as do the eigenvalues of $H(N,W)$. The remainder of this paper will concentrate on DPSS generation from the tridiagonal matrix $T(N,W)$.

Using the QR algorithm or a variation of it generally requires the entire set of eigenvectors to be found. In various possible applications of DPSS's, only a select number of DPSS's will be desired. In these cases, it would be beneficial to have a different algorithm which finds only a single DPSS. The following describes a procedure for finding a single DPSS (eigenvector) associated with the kth largest eigenvalue of normal symmetric Jacobian matrices such as $T(N,W)$. In general, the algorithm first finds the desired eigenvalue, and the corresponding eigenvector is found using iterative techniques.

Because $T(N,W)$ is originally in symmetric tridiagonal form, the well known method of bisection may be used to find a single eigenvalue with potentially smaller relative error than if multiple eigenvalues were found using QR or QL iterations (p. 439 of [9]). Let the elements of T be denoted by

$$T = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & 0 & \vdots \\ 0 & & & & \\ \vdots & & & & b_{n-1} \\ 0 & & \cdots & b_{n-1} & a_n \end{bmatrix}.$$

The characteristic polynomial of the leading $r \times r$ principal submatrix, T_r, of T is known to be

$$p_r(x) = \det(T_r - xI) \quad r = 1,2,\ldots,n. \tag{4}$$

If we set $p_0(x) = 1$ and $p_1(x) = a_1 - x$, then a simple determinantal expansion (p. 437 of [9]) reveals the recursive relationship

$$p_r(x) = (a_r - x)p_{r-1}(x) - b_{r-1}^2 p_{r-2}(x) r = 2,\ldots,n. \tag{5}$$

The roots of $p_n(x)$ are the eigenvalues of T, and bisection may now be used to find any or all of the roots. However, finding a specified eigenvalue, say the kth largest, also requires the use of a theorem called the Sturm sequence property (p. 438 of [9]). The portion of the theorem to be used here states that the number of sign

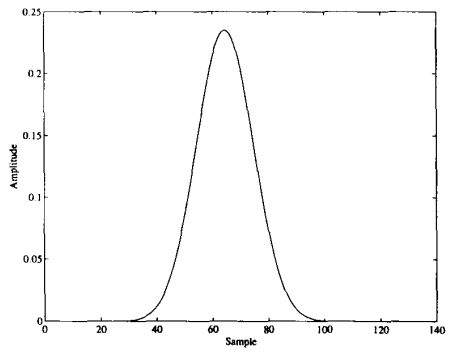

Fig. 1. DPSS $v^{(0)}(N = 128, W = .10)$.

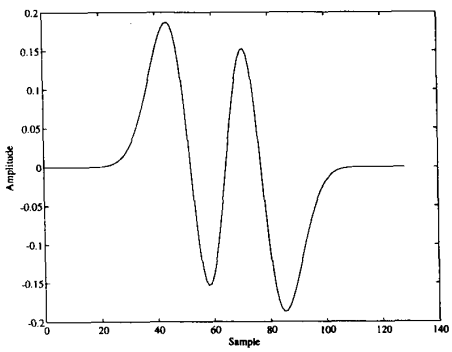

Fig. 2. DPSS $v^{(3)}(N = 128, W = .10)$.

changes $a(\lambda)$ in the sequence

$$\{p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)\}$$

equals the number of eigenvalues that are less than λ.

Now the kth largest eigenvalue may be found with an accuracy of **TOL** using the following algorithm:

```
while |λmax − λmin| > TOL
    λk = (λmax − λmin)/2
    if a(λk) ≥ k
        λmax = λk
    else
        λmin = λk
    end
end.
```

Of course, initially the interval $[\lambda_{\min}, \lambda_{\max}]$ must contain the desired eigenvalue. The interval in which all eigenvalues were contained for a matrix T of size N was found using the Gershgorin circle theorem (p. 341 of [9] to be $\lambda \in [-y, y]$, where

$$y = \frac{(N-1)*(N+2) + N*(N+1)}{8} + 0.25.$$

Because the notation used by Slepian associates the most compact DPSS and largest eigenvalue with the index value $k = 0$ instead of $k = N-1$, this reverse ordering must be accounted for when searching for a specific eigenvalue.

Once the desired eigenvalue is found, the corresponding eigenvector or DPSS may be found using the method of inverse iteration [10]. Inverse iteration works fine for finding the eigenvector which corresponds to the known eigenvalue except when the eigenvalue is close in value to another eigenvalue, i.e., the eigenvalues are clustered. The procedure for inverse iteration is shown below for finding the eigenvector associated with eigenvalue λ_k.

let v_1 be a unit vector

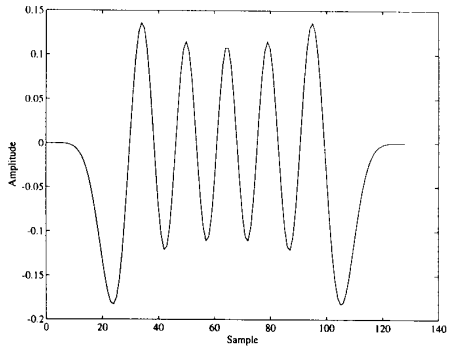

Fig. 3. DPSS $v^{(10)}(N = 128, W = .10)$.

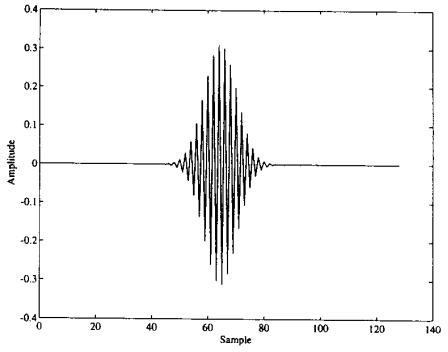

Fig. 4. DPSS $v^{(127)}(N = 128, W = .10)$.

for $i = 1, 2, \ldots$
Solve $(T - \lambda_k I)v_2 = v_1$ for v_2
Normalize v_2 to get a new v_1
end.

The desired DPSS is the last normalized solution, v_1. Even though $(T-\lambda_k I)$ is nearly singular for eigenvalues λ_k of very good accuracy, inverse iteration is known to work well in these cases, many times requiring only one or two iterations [10]. The norm of the residue $r = (T - \lambda_k I)v_1$ may be used to determine the accuracy of the final result. When the known eigenvalue λ_k was accurate to within $\mathbf{TOL} = 1e - 6$ using the bisection method explained previously, it was found that three iterations of the above procedure produced accurate DPSS's. Because matrix $T(N, W)$ is tridiagonal, solving the system equations during each iteration for the latest eigenvector estimate was also simplified (p. 155 of [9]).

III. Some Examples

Figs. 1–5 show some examples of DPSS's generated using the technique described above.

IV. Conclusion

A method for accurate and computationally efficient generation of any single DPSS of large length was presented. The method is easy to implement and should prove useful in situations where only a few DPSS's out of a large set are desired. The availability of long length DPSS's will open the oportunity for their use in applications requiring sequences of long length.

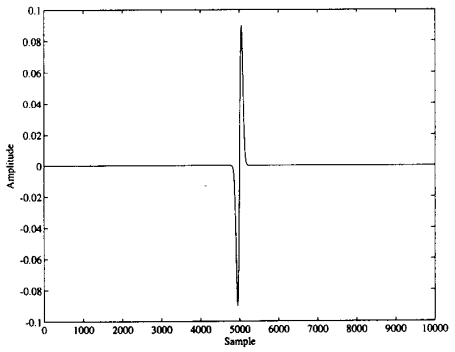

Fig. 5. DPSS $v^{(1)}(N = 10000, W = .10)$.

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Noniterative and Fast Iterative Methods for Interpolation and Extrapolation

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***Abstract*—In this correspondence we study the band-limited interpolation and extrapolation problems for finite-dimensional signals. We show that these problems can be easily reduced to the solution of a set of linear equations with a real symmetric positive-definite matrix S with spectral radius $\rho(\mathbf{S}) < 1$. Thus, the equations can be solved directly or using successive approximation methods. A number of other well known methods which may substantially increase the convergence rate may also be readily applied and are briefly discussed. We state conditions for their convergence, and illustrate their performance through an example.**

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